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On the Strength of Einstein's Unified Field Equations

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Einstein's concept of the "strength" of a system of field equations, which has been related in a simple way to the amount of initial data required for the system, is examined for his last unified field theory. The apparently surprising weakness of this system is traced to the high order of the associated electromagnetic field equations. These equations allow the existence of purely electric longitudinal waves, in spite of the absence of any "photon mass."

In his last work on unified field theory, Einstein (1955) introduced a way of quantifying what he called the "strength" of a system of field equations. The field components are expanded in Taylor series, and the number of nth-order coefficients left free by the field equations, with proper accounting for identities and gauge freedom, is calculated. The asymptotic value of this quantity is of the form

$$z \sim [4 \cdot 5 \ldots (n+3)/1 \cdot 2 \ldots n](z_1/n) \tag{1}$$

z_1 was called the "coefficient of freedom" by Einstein. For the scalar wave equation, $z_1 = 6$, while $z_1 = 12$ both for Maxwell's vacuum equations and for Einstein's vacuum gravitational equations. On the other hand, $z_1 = 42$ for the nonsymmetric, mixed affine–metric field equations which Einstein proposed as a unified field theory.

The precise significance of this "coefficient of freedom" remained unclear for some time. Finally, Schutz (1975) showed that $z_1/3$ is the number of free initial data which can be specified for the system at each point of a spacelike hypersurface. The results mentioned above are thus in agreement with the well-known facts that the scalar wave equation describes a physical system with one degree of freedom, requiring two initial

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data per space point, while Maxwell's and Einstein's gravitational equations each describe systems with two degrees of freedom per space point. Other systems of field equations have been considered by Mariwalla (1974) and by Burman (1977).

The result $z_1 = 42$ for Einstein's nonsymmetric theory has, however, remained unexplained. This result might suggest that the fields described by this theory possess a total of seven degrees of freedom per space point, prompting speculation about photon or graviton masses, or about new types of fields in addition to the gravitational and electromagnetic ones.

It will be shown here that this theory's required 14 pieces of initial data per space point (rather than the 8 which would be needed for the uncoupled Einstein–Maxwell equations) are due to the higher differential order of the new field equations. This will be shown by means of an analysis of the weak-field limit of the system, a limit which should introduce no specialization as far as the number of degrees of freedom of the system is concerned. In this limit, the only new wave mode which is introduced is a longitudinal electric wave.

The field equations which we wish to consider can be put into the form

$$^{*}R_{(\kappa\lambda)} = 0 \tag{2a}$$

$$^{*}R_{[\kappa\lambda],\iota} + {}^{*}R_{[\lambda\iota],\kappa} + {}^{*}R_{[\iota\kappa],\lambda} = 0 \tag{2b}$$

$$g_{\mu\nu,\lambda} - {}^{*}\Gamma^{\kappa}_{\mu\lambda} g_{\kappa\nu} - {}^{*}\Gamma^{\kappa}_{\lambda\nu} g_{\mu\kappa} = 0 \tag{2c}$$

with $^{*}R_{\mu\nu}$ the Ricci tensor of Schrödinger's (1963) star affinity,

$$^{*}\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \left(\tfrac{1}{3}\right)\delta^{\lambda}_{\mu}(\Gamma^{\kappa}_{\nu\kappa} - \Gamma^{\kappa}_{\kappa\nu})$$

Here square and round brackets indicate, respectively, antisymmetrization and symmetrization.

An important consequence of the preceding equations is

$$\mathfrak{g}^{[\mu\nu]}{}_{,\nu} = 0 \tag{3}$$

where $\mathfrak{g}^{\mu\nu}$ is a contravariant tensor density associated with the nonsymmetric field $g_{\mu\nu}$.

The set of equations (2c) has been solved by Hlavatý (1957). We shall require here only the solution through terms linear in the fields $k_{\mu\nu} \equiv g_{[\mu\nu]}$ and $h_{\mu\nu} \equiv g_{(\mu\nu)} - \eta_{\mu\nu}, \eta_{\mu\nu}$ being the metric tensor of flat space–time. This approximate expression for the star affinity is

$$^{*}\Gamma^{\nu}_{\lambda\mu} \simeq \left\{ {}_{\lambda}{}^{\nu}{}_{\mu} \right\} + \tfrac{1}{2}\left(k^{\nu}{}_{\mu,\lambda} + k_{\lambda}{}^{\nu}{}_{,\mu} + k_{\lambda\mu,}{}^{\nu}\right) \tag{4}$$

where $\{{}_{\lambda}{}^{\nu}{}_{\mu}\}$ is the Christoffel affinity formed from $g_{(\mu\nu)}$, to first order in $h_{\mu\nu}$. Indices are raised and lowered with $\eta_{\mu\nu}$, in accord with our linear approximation.

The set of equations (2a) is just the statement that the linearized Ricci tensor formed from $g_{(\mu\nu)}$ must vanish. Thus we have the linearized gravitational field equations for vacuum, which require four pieces of data per space point for their initial-value problem.

Equation (3) becomes, in this approximation,

$$k_{\mu\nu}{}^{,\nu}=0 \tag{5}$$

and this result allows us to reduce equation (2b) to the form

$$\Box\left(k_{\kappa\lambda,\mu}+k_{\lambda\mu,\kappa}+k_{\mu\kappa,\lambda}\right)=0 \tag{6}$$

For a count of the initial data required for equations (5) and (6), the precise way in which the electromagnetic field $F_{\mu\nu}$, made up of the 3-vectors $\mathbf{E}$ and $\mathbf{B}$ in the usual way, is identified with components of the skew-symmetric tensor $k_{\mu\nu}$ is unimportant. Here I shall treat $k_{\mu\nu}$ as the dual of $F_{\mu\nu}$ (e.g., Murphy, 1975). The set (5) then yields half of Maxwell's vacuum equations,

$$\nabla\cdot\mathbf{B}=0 \tag{7a}$$

and

$$\partial\mathbf{B}/\partial t=-\nabla\times\mathbf{E} \tag{7b}$$

The set (6), however, will give us only the d'Alembertian of the other half of Maxwell's equation. We can write this set in the form

$$\nabla\cdot\partial^2\mathbf{E}/\partial t^2=\nabla^2(\nabla\cdot\mathbf{E}) \tag{8a}$$

$$\partial^3\mathbf{E}/\partial t^3-\nabla\times\partial^2\mathbf{B}/\partial t^2-\nabla^2\partial\mathbf{E}/\partial t=-\nabla^2(\nabla\times\mathbf{B}) \tag{8b}$$

[Alternatively, the time derivative of $\mathbf{B}$ in (8b) may be eliminated by means of (7b).]

We must now count the data which must be specified at each point of the spacelike hypersurface $t=0$ for the analytic Cauchy problem. Three components of $\mathbf{B}$ are subject to the constraint (7a), as in Maxwellian electrodynamics, leaving two independent components of the magnetic induction to be chosen freely. Three components of $\mathbf{E}$ can also be chosen freely, and $\partial\mathbf{B}/\partial t$ will then be given by (7b).

We now consider the equation (8b). This involves second time derivatives of $\mathbf{B}$, and third time derivatives of those of $\mathbf{E}$. $\mathbf{B}$ and $\partial\mathbf{B}/\partial t$ are already taken care of in our previous count, and $\mathbf{E}$ is also given. In addition, we must specify the six components of $\partial\mathbf{E}/\partial t$ and $\partial^2\mathbf{E}/\partial t^2$ in order to obtain a unique solution of (8b).

However, the components of $\partial^2\mathbf{E}/\partial t^2$ cannot be given arbitrarily, for they must satisfy the constraint (8a). Thus a total of five functions will give the required time derivatives of the electric field on the initial-value hypersurface. These, together with the three components of $\mathbf{E}$ and the two free components of $\mathbf{B}$, make up ten pieces of electromagnetic data per space point. The gravitational degrees of freedom bring the total data per space point for the original system of field equations to the desired value $14 = z_1/3$.

As one would expect, there are solutions to the system of equations (7) and (8) which are not shared by Maxwell's equations. If we look for solutions of the form

$$\mathbf{E} = \mathbf{E}_0 \exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)], \qquad \mathbf{B} = \mathbf{B}_0 \exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)]$$

then we find from (7) and (8) that we must require

$$\mathbf{k}\cdot\mathbf{B}_0 = 0 \tag{9a}$$

$$\mathbf{k}\times\mathbf{E}_0 = \omega\mathbf{B}_0 \tag{9b}$$

$$(\mathbf{k}\cdot\mathbf{E}_0)(\omega^2 - k^2) = 0 \tag{9c}$$

$$\omega^{-1}(\omega^2 - k^2)\left[\mathbf{E}_0(\omega^2 - k^2) + \mathbf{k}(\mathbf{k}\cdot\mathbf{E}_0)\right] = 0 \tag{9d}$$

if $\mathbf{B}_0$ is eliminated from the final equation by means of (9b).

The first two equations of this set tell us that, as usual, the magnetic induction must be orthogonal both to the propagation vector and to the electric field. (9c) then states that either the electric field is orthogonal to the propagation vector *or* the phase velocity of the waves is unity. If we chose $\mathbf{k}\cdot\mathbf{E}_0 = 0$, (9d) then tells us that we will also have $k^2 = \omega^2$ in all nontrivial cases. However, $k^2 = \omega^2$ ensures that both (9c) and (9d) will be satisfied without any other conditions. Thus it is not necessary that $\mathbf{k}$ and $\mathbf{E}_0$ be orthogonal, and longitudinal electric waves can exist.

These longitudinal electric waves (with which no magnetic induction need be associated) travel at unit speed, and the theory here is, in general, rather different from theories with a photon mass in which longitudinally polarized waves appear. [See, for example, Murphy and Burman (1978) for observational limits on such theories.] Nevertheless, the existence of such longitudinal waves suggests, in principle, observational tests of Einstein's unified field theory. What is now required is a model for the coupling of the electromagnetic field to charges which would allow the intensity of emitted longitudinal waves to be calculated for this theory.

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